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Evolution equations and Lévy processes on quantum groups

U Franz†‡ and R Schott†

† Institut Elie Cartan and CRIN-CNRS, BP 239, Université H. Poincaré-Nancy I, F-54506 Vandœuvre-lès-Nancy, France

‡ Arnold Sommerfeld Institut für math. Physik, Technische Universität Clausthal, Leibnitzstr. 10, D-38678 Clausthal-Zellerfeld, Germany

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Abstract. Evolution equations like the heat or diffusion equation or the Schrödinger equation can be associated with stochastic processes. In this paper we study the relation between equations of the form $\partial_t u = Lu$ and Lévy processes (i.e. quantum stochastic processes with independent and stationary increments) on quantum groups and braided groups. Solutions of such equations are calculated as Appell systems. Wigner distributions of these processes are defined and it is proven that they satisfy a Fokker–Planck equation.

1. Introduction

Quantum groups are unital associative algebras, equipped with an additional structure, the coalgebra structure, that allows one to define notions like increments, translations, etc, and thus allows one to define analogues of many (physically!) important concepts for stochastic processes on vector spaces or groups, as, for example, the notion of stationary and independent increments that is at the basis of Brownian motion and diffusions, or the various characterizations of Gauss distributions. We recommend recent books by Majid [13] (in particular, ch 5), Meyer [14] and Schürmann [15] as an introduction to this field (see also [16]).

In this paper we consider stochastic processes on quantum groups that are related to evolution equations of the form

$$\partial_t u = Lu$$

with some difference-differential operator L. For the equations considered in section 3, u is an element of a quantum or braided group $\mathcal{A}$. We recall that solutions of these equations can be given as Appell systems or shifted moments of the associated process, and show how these can be calculated explicitly on the q-affine group, the braided line and plane, and a braided analogue of the Heisenberg–Weyl group. These calculations are original.

In section 4, which is the main contribution of this paper, we define a Wigner map from functionals on a quantum group or braided group to a 'Wigner' density on the undeformed space. We prove that the densities associated in this way to Lévy processes (i.e. processes with independent and stationary increments) satisfy a Fokker–Planck-type equation. In the one-dimensional case these coincide with the evolution equations of section 3, but in the general case we get new equations.

We close with a few final remarks in section 5.

2. Preliminaries

Notation. Let $q \in \mathbb{C}$. We set $q_n = \sum_{\nu=0}^{n-1} q^\nu$, i.e. $q_n = n$ if $q = 1$, and $q_n = (q^n - 1)/(q-1)$ otherwise. We will also use $q_n! = \Pi_{\nu=1}^{n} q_\nu$, the q-exponential series $\mathrm{e}_q^x = \sum_{n=0}^{\infty} x^n/q_n!$ (if q is not a root of unity), and the q-binomial coefficients defined by the recurrence relation

$$\begin{bmatrix} m+1 \\ \mu \end{bmatrix}_q = \begin{bmatrix} m \\ \mu \end{bmatrix}_q + q^{m-\mu+1} \begin{bmatrix} m \\ \mu - 1 \end{bmatrix}_q \begin{bmatrix} m \\ 0 \end{bmatrix}_q = \begin{bmatrix} m \\ m \end{bmatrix}_q = 1.$$

If q is not a root of unity one has

$$\begin{bmatrix} m \\ \mu \end{bmatrix}_q = \frac{q_m!}{q_\mu! \cdot q_{m-\mu}!}.$$

For the definition of quantum groups and braided groups (or Hopf algebras and braided Hopf algebras) see [13]. We recall a few examples which we shall use to illustrate our approach. For the origin of the first two examples see [11, 12], the third is well known in the quantum group literature, while it seems that the last was first studied (as a braided Hopf algebra) by the present authors [8] (but see also the appendix of [2]).

The braided line $\mathbb{R}_q$. Let $q \in \mathbb{C}$, q not a root of unity. The braiding $\Psi(x^n \otimes x^m) = q^{nm} x^m x^n$ turns the algebra of polynomials in one variable into a braided Hopf algebra (with $\Delta x = x + x'$). This Hopf algebra, denoted by $\mathbb{R}_q$, can be dually paired with itself, set $\langle p^n, x^m \rangle = q_n! \delta_{nm}$. The dual copy acts on $\mathbb{R}_q$ via $\rho(u)a = \sum \langle u, a^{(1)} \rangle a^{(2)}$, where $\Delta a = \sum a^{(1)} \otimes a^{(2)}$. One finds that $\rho(p)$ is the q-difference operator $\delta_q : f(x) \mapsto (f(qx) - f(x))/x(q-1)$.

The braided plane $\mathbb{C}_q^{2|0}$. The braided plane is the braided Hopf algebra with two q-commuting primitive generators x, y and braid relations $x'x = q^2 xx'$, $x'y = qyx'$, $y'x = qxy' + (q^2 - 1)yx'$, $y'y = q^2 yy'$.

$\mathbb{C}_q^{2|0}$ is dually paired with $\mathbb{C}_{q^{-1}}^{2|0}$ and thus there are partial derivatives,

$$\partial_1 x^n y^m = (q^2)_n x^{n-1} y^m \qquad \partial_2 x^n y^m = (q^2)_m q^n x^n y^{m-1}.$$

The q-affine group Aff_q. Let $\alpha, \beta \in \mathbb{C}$, st. $q = \mathrm{e}^{\alpha\beta}$ is not a root of unity. The q-affine group Aff_q is the Hopf algebra with two generators a, b and relations

$$ba = (a + \beta)b \qquad \Delta a = a + a' \qquad \Delta b = b + \mathrm{e}^{\alpha a} b'$$

and trivial braiding, i.e. $\Psi = \tau$ (the twist map). We will denote the generators of the dual of Aff_q, i.e. $U_q(\mathrm{Aff})$, by X and Y, and the dual action by ρ.

The braided q-Heisenberg-Weyl group HW_q. We give a braided Hopf algebra structure for the algebra known as the q-oscillator algebra or q-Heisenberg–Weyl algebra, i.e. the algebra defined by $ac - qca = \mathbb{1}$, $\mathbb{1}$ central. We can regard this algebra as generated by two generators a and c with the cubic relations

$$aac + qcaa = (1+q)aca \qquad acc + qcca = (1+q)cac$$

$q \in \mathbb{C}$, q not a root of unity. If we define the braiding by $a'a = qaa'$, $a'c = ca'$, $c'a = (1/q)ac'$, $c'c = qcc'$, then $\Delta a = a + a'$, $\Delta c = c + c'$ defines an algebra homomorphism from HW_q to $\mathrm{HW}_q \tilde{\otimes} \mathrm{HW}_q$. The braided bialgebra defined in this way also admits an antipode, define S by $S(a) = -a$, $S(c) = -c$, and extend via $S \circ m = m \circ \Psi \circ (S \otimes S)$.

If we introduce a third generator, for example by $ac - qca = (1-q)b$ or $d = ac - ca$, then $\{a^n b^m c^r; n, m, r \in \mathbb{N}\}$ and $\{a^n d^m c^r; n, m, r \in \mathbb{N}\}$ are bases (Poincaré–Birkhoff–Witt bases) of HW_q (b is central, and d satisfies $ad = qda$ and $dc = qcd$).

A *-structure is defined by $a^* = c$, $c^* = a$, if q is real.

If we define partial differential operators on HW_q via

$$\rho(x)a = 1 \qquad \rho(x)c = 0$$
$$\rho(z)a = 0 \qquad \rho(z)c = 1$$

and extend with the Leibnitz rules

$$\rho(x)(au) = u + qa\rho(x)u \qquad \rho(x)(cu) = (1/q)c\rho(x)u$$
$$\rho(z)(au) = a\rho(z)u \qquad \rho(z)(cu) = u + qc\rho(z)u$$

for $u \in \mathrm{HW}_q$, then $\rho(x)$ and $\rho(z)$ satisfy again the HW_q-relations, and $\langle X, u\rangle = \varepsilon(\rho(X)u)$ defines a dual pairing.

Quantum stochastic processes. A quantum probability space is usually defined as a pair $(\mathcal{A}, \Phi)$ consisting of a *-algebra $\mathcal{A}$ and a state (i.e. a normed positive linear functional) Φ on $\mathcal{A}$. A quantum random variable j over a quantum probability space $(\mathcal{A}, \Phi)$ on a *-algebra $\mathcal{B}$ is simply a *-algebra homomorphism $j : \mathcal{B} \to \mathcal{A}$. A quantum stochastic process is a family of quantum random variables over the same quantum probability space and taking values in the same algebra.

Here we focus on processes with independent increments indexed by $\mathbb{R}_+$; they are characterized by their one-dimensional distributions $\{\varphi_t; t \in \mathbb{R}_+\}$. If the increments are also stationary, then the one-dimensional distributions form a convolution semi-group, i.e. $\varphi_0 = \varepsilon$ and $\varphi_s \star \varphi_t = \varphi_{s+t}$. Such processes are called white noise or Lévy processes [15].

In this case the transition operators associated with the process are defined by

$$U_t(\varphi) : a \mapsto \sum \varphi_t(a^{(1)})a^{(2)} \qquad \text{if } \Delta a = \sum a^{(1)} \otimes a^{(2)}.$$

These operators form a semi-group, $U_s(\varphi) \circ U_t(\varphi) = U_{s+t}(\varphi)$, and $U_0(\varphi) = \mathrm{id}$.

We will say that φ is associated with the equation $(\partial_t - L)u = 0$, if $U_t(\varphi) = \mathrm{e}^{tL}$, and call L the generator of φ in this case.

3. Appell systems

We will consider equations of the form

$$\partial_t u = Lu \tag{1}$$

where $L : \mathcal{A} \to \mathcal{A}$ is a differential operator, independent of t, for example

$$\partial_t u = (a\delta_q^2 + b\delta_q)u \qquad \text{on } \mathbb{R}_q$$
$$\partial_t u = (\partial_1^2 + \partial_2^2)u \qquad \text{on } \mathbb{C}_q^{2|0}$$
$$\partial_t u = (\rho(X)^2 + \rho(Y)^2)u \qquad \text{on } \mathrm{Aff}_q$$
$$\partial_t u = (\rho(x)^2 + \rho(z)^2)u \qquad \text{on } \mathrm{HW}_q.$$

In the first equation, L is a general second-order q-difference operator, but for the explicit calculations we shall assume that a and b are constants.

In the second equation we have an analogue of the Laplacian, the operator in the third equation is related to the Gegenbauer or ultraspherical polynomials, see for example [6]. In the fourth equation we have an analogue of the Kohn–Laplacian on the Heisenberg group.

An equation of the form (1) gives rise to a transition operator, formally written as e^{tL}. If the functional $\varepsilon \circ L$ is conditionnally positive (with respect to an involution on $\mathcal{A}$), then $\varepsilon \circ e^{tL}$ defines a convolution semi-group of states and thus a Lévy process (if the braiding is different from the twist map, then we have to impose the additional condition that $\varepsilon \circ L$ is Ψ-invariant, i.e. that $(\varepsilon \circ L \otimes \phi) \circ \Psi = \phi \otimes \varepsilon \circ L$ for all $\phi \in \mathcal{A}^*$). We can still associate a process with L, even if $\varepsilon \circ L$ is not conditionally positive, but in this case the state fails to be positive, see [3].

Appell systems on Lie groups have been studied in [4], quantum groups were considered in [3]. The results presented here for braided groups are original. We recall the definition of Appell systems, see [3, 7].

Definition 1. We define the (left) Appell polynomials on a braided group $\mathcal{A}$ with respect to a semi-group of functionals $\{\varphi_t\}$ by

$$h_k = (\varphi_t \otimes \mathrm{id}) \circ \Delta a_k$$

i.e. $h_k = U_t(\varphi)(a_k)$, for a fixed basis $\{a_k\}$ of $\mathcal{A}$.

If L is the generator of $\{\varphi_t\}$, then h_k solves

$$\partial_t h_k = L h_k.$$

For other interesting properties, for example, raising operators, or in relation to matrix elements, see [3, 7].

3.1. *Example: the braided line*

The Appell polynomials associated with the functional $\varphi_t = \exp(tL)$ where

$$L = \frac{a^2}{1+q} p^2 + bp$$

are

$$\begin{aligned} h_k(x;t) = (\varphi_t \otimes \mathrm{id}) \circ \Delta x^k &= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q \varphi_t(x^\nu) x^{k-\nu} \\ &= \sum_{\nu=0}^{k} \frac{q_k! H_\nu(bt, -2a^2 t/(1+q))}{q_{k-\nu}! \nu!} x^{k-\nu} \end{aligned}$$

where H_ν denotes the Hermite polynomials, defined by

$$H_p(x,t) = \sum_{k=0}^{[p/2]} \binom{p}{2k} \frac{(2k)!}{2^k k!} x^{p-2k} (-t)^k.$$

These Appell polynomials are solutions of

$$\partial_t u = \frac{a^2}{1+q} \delta_q^2 u + b \delta_q u.$$

For $b = 0$, $a = \sqrt{(1+q)/2}$ the Appell polynomials simplify to

$$h_k(x;t) = \sum_{\nu=0}^{[k/2]} \frac{q_k! t^\nu x^{k-2\nu}}{q_{k-2\nu}! 2^\nu \nu!}.$$

These polynomials are a q-analogue of the Hermite polynomials.

3.2. Example: the braided plane

We choose $\phi_t = \exp tL$ with

$$L = \frac{1}{1+q^2}(\partial_1^2 + \partial_2^2).$$

We get

$$\phi_t = \sum_{r=0}^{\infty} \frac{t^r}{(1+q^2)^r r!} \sum_{\nu=0}^{r} \begin{bmatrix} r \\ \nu \end{bmatrix}_{q^{-4}} \partial_2^{2\nu} \partial_1^{2(r-\nu)}$$

since $\partial_2\partial_1 = q^{-1}\partial_1\partial_2$. This leads to the following formula for the Appell polynomials:

$$\begin{aligned} h_{rm}(x, y; t) &= \exp(tL)x^r y^m \\ &= \sum_{\nu,\mu=0}^{[r/2],[m/2]} \frac{\left[{\mu+\nu \atop \nu}\right]_{q^{-4}} (q^2)_r!(q^2)_m! q^{2\mu(r-2\nu)} t^{\mu+\nu}}{(q^2)_{r-2\nu-1}!(q^2)_{m-2\mu-1}!(1+q^2)^{\mu+\nu}(\mu+\nu)!} x^{r-2\nu} y^{m-2\mu}. \end{aligned}$$

These polynomials solve the evolution equation

$$\partial_t u = \frac{1}{1+q^2}(\partial_1^2 + \partial_2^2)u$$

where ∂_1, ∂_2 can be defined by

$$\partial_1 f(x, y) = \frac{f(q^2x, y) - f(x, y)}{x(q^2-1)}$$

$$\partial_2 f(x, y) = \frac{f(qx, q^2y) - f(qx, y)}{y(q^2-1)}.$$

3.3. Example: the q-affine group

We can use the generalized Gegenbauer polynomials defined in [6]

$$C_n^h(x) = \sum_{\nu=0}^{[n/2]} \frac{(h)_{n-\nu}}{q_{n-2\nu}} \frac{(-1)^\nu}{\nu!} (2x)^{n-2\nu}$$

and the representation $\rho_h(X) = \alpha(x\partial_x + h)$, $\rho_h(Y) = \mathrm{i}\alpha\delta_x$ to calculate the moments of $\Phi_t = \exp(\frac{1}{2}t(X^2+Y^2))$. These polynomials are eigenfunctions of

$$S_h = (x\partial_x + h)^2 - \delta_x^2 = \rho_h(X)^2 + \rho_h(Y)^2 \qquad \text{i.e. } S_h C_n^h(x) = (n+h)^2 C_n^h(x)$$

and their inversion formula is

$$x^n = \frac{q_n!}{2^n} \sum_{k=0}^{[n/2]} \frac{h+n-2k}{(h)_{n-k+1}k!} C_{n-2k}^h(x).$$

Using the Feynman–Kac-type formula (cf [3])

$$\Phi_t(\mathrm{e}^{a\rho_h(X)}\, \mathrm{e}_q^{b\rho_h(Y)} x^n) = \mathrm{e}^{\frac{1}{2}t(\rho_h(x)^2 + \rho_h(Y)^2)} x^n$$

and comparing the coefficients of x^{n-2r} we get

$$\Phi_t(\mathrm{e}^{(n-2r+h)\alpha a} b^{2r}) = \frac{q_{2r}}{\alpha^{2r}} \sum_{k=0}^{r} \frac{(h+n-2k)(h)_{n-r-k}(-1)^k}{4^r (h)_{n-k+1} k!(r-k)!}\, \mathrm{e}^{(n-2k+h)^2\alpha^2 t/2} \qquad \text{for } n \geqslant 2r$$

$$\Phi_t(\mathrm{e}^{(n-2r+h)\alpha a} b^{2r+1}) = 0 \qquad \text{for } n \geqslant 2r+1.$$

Differentiating ν times with respect to h and setting $h = m - \mu + 2r - n$, we obtain all moments that are needed to calculate the Appell functions

$$h_{nm}(a, b, t) = \Phi_t(\Delta(a^n b^m)) = \sum_{\nu=0}^{n} \sum_{\mu=0}^{m} \binom{n}{\nu} \begin{bmatrix} m \\ \mu \end{bmatrix}_q \Phi_t(a^\nu \, \mathrm{e}^{(m-\mu)\alpha a} b^\mu) a^{n-\nu} b^{m-\mu}.$$

3.4. Example: the braided q-Heisenberg–Weyl group

Consider $L = x^2 + z^2$. Then

$$\mathrm{e}^{tL} = \sum_{\nu=0}^{n} \sum_{\kappa=0}^{2(n-\nu)\wedge 2\nu} \frac{C^n_{\nu,\kappa} t^n}{n!} z^{2(n-\nu)-\kappa} y^\kappa x^{2\nu-\kappa}$$

where the coefficients $C^n_{\nu,\kappa}$ are determined by the recurrence relations

$$C^{n+1}_{\nu,\kappa} = C^n_{\nu-1,\kappa} + q^{2\kappa} C^n_{\nu,\kappa} + q_{2\nu-\kappa+1} q_2 q^{\kappa-1} C^n_{\nu,\kappa-1} + q_{2\nu-\kappa+2} q_{2\nu-\kappa+1} C^n_{\nu,\kappa-2}.$$

For $\kappa = 0$ we have the binomial coefficients $C^n_{\nu,0} = \binom{n}{\nu}$. This allows us to calculate

$$h_{nmr}(t) = \mathrm{e}^{tL}(a^n b^m c^r)$$

using, for example, the dual pairing

$$\langle z^r y^m x^n, a^{n'} b^{m'} c^{r'} \rangle = \delta_{nn'} \delta_{mm'} \delta_{rr'} q_n! q_m! q_r!.$$

4. Densities

For one single variable, or in the commutative case, one can use Bochner's theorem to associate a density to a quantum random variable, cf [14].

We now want to associate joint densities to several non-commutating variables, along the line of Wigner distributions [17]. We will map functionals on an algebra with n generators to measures on $\mathbb{R}^n$. Equivalently, we can ask for a map from functions on $\mathbb{R}^n$ (e.g., polynomials) to elements of the algebra.

Consider the following diagram:

$$\begin{array}{ccccc} & & \text{Wigner} & & \\ & \text{QS} & \longrightarrow & \text{CS} & \\ \text{Duality} & \updownarrow & & \updownarrow & \text{Duality} \\ & \text{QO} & \longleftarrow & \text{CO} & \\ & & \text{Weyl} & & \end{array}$$

where QS is (the linear span of) the set of quantum states, CS is (the linear span of) the set of classical states, QO is the set of quantum observables and CO is the set of classical observables.

We want the following similar diagram:

$$\begin{array}{ccccc} & & \text{`Wigner'} & & \\ & \mathcal{U} & \longrightarrow & \mathcal{M}(X) & \\ q\text{-Fourier} & \updownarrow & & \updownarrow & \text{Fourier} \\ & \mathcal{A} & \longleftarrow & C(X) & \\ & & \text{`Weyl'} & & \end{array}$$

where X is the undeformed space or group, and $\mathcal{M}(X)$ denotes the (convolution) algebra of (signed) measures on X, $C(X)$ the algebra of continuous functions on X, and $\mathcal{A}$ and $\mathcal{U}$

the quantum group and algebra, respectively. The q-Fourier transformation (with respect to an integral $\int_{\mathcal{A}}$ on $\mathcal{A}$) is defined by

$$\mathcal{F}_{\mathcal{A}}(u) = \int_{\mathcal{A}} (u \exp)$$

where exp is the exponential or coevalution map of $\mathcal{A}$ and $\mathcal{U}$, see [9, 10]. We have $\int_{\mathcal{A}}(ab) = \langle \mathcal{F}_{\mathcal{A}}(a), b\rangle$, and thus in this setting a density (with respect to $\int_{\mathcal{A}}$) of a functional $\Phi \in \mathcal{A}^*$ can, at least in principle, be calculated with the inverse Fourier transform, $\rho_\Phi = \mathcal{F}_{\mathcal{A}}^{-1}(\Phi)$. A more detailed discussion, including an explicit example on $\mathbb{R}_q$, can be found in [7].

Here we shall use the right-hand side of the diagram to introduce densities that 'live' on the classical, i.e. undeformed, group or space. Following Anderson [1], we fix a set of generators $x_1, \ldots, x_n$ of $\mathcal{A}$ and define the Weyl map [16] on polynomials $u_1^{k_1} \ldots u_n^{k_n}$ by

$$W(u_1^{k_1} \ldots u_n^{k_n}) = \frac{k_1! \ldots k_n!}{k!} \sum_{\pi \in \mathcal{S}_k} x_{\pi(1)} \ldots x_{\pi(k)}$$

where $k = k_1 + \cdots + k_n$. Other definitions are possible, for example $W_W : u_1^{k_1} \ldots u_n^{k_n} \mapsto x_1^{k_1} \ldots x_n^{k_n}$ ('Wick'), or $W_{AW} : u_1^{k_1} \ldots u_n^{k_n} \mapsto x_n^{k_n} \ldots x_1^{k_1}$ ('anti-Wick'), or also $W_{q-\exp}$ defined by $\mathrm{e}^{u\cdot v} \mapsto \exp(x|v)$. However, $W_{q-\exp}$ will not leave the marginal distributions unchanged. In fact, W is uniquely determined by the conditions $W(u_i) = x_i$ and

$$W((a_1 u_1 + \cdots + a_n u_n)^k) = (a_1 W(u_1) + \cdots + a_n W(u_n))^k$$

and thus only W^* will give the correct marginal distributions for all linear combinations of the generators. Ordered monomials like the 'Wick' or 'anti-Wick' map still lead to the right marginal distributions for the generators.

The Wigner map W^* is defined by the condition

$$\langle \Phi, W(u)\rangle = \int u \, \mathrm{d}W^*(\Phi)$$

i.e. as the dual of the Weyl map. The Fourier transform of the measure $W^*(\Phi)$ is

$$g_\Phi(v) = \mathcal{F}(W^*(\Phi))(v) = \int \mathrm{e}^{\mathrm{i}u\cdot v} \, \mathrm{d}W^*(\Phi) = \langle \Phi, W(\mathrm{e}^{\mathrm{i}u\cdot v})\rangle$$

where we have assumed that we can interchange the limits involved, and that this series defines an analytic function. Note that here $\int$ and $\mathcal{F}$ denote integration and the Fourier transform on X, respectively.

If the functionals Φ_t form a convolution semigroup, then the associated Wigner densities satisfy an evolution equation or Fokker–Planck equation.

Proposition 2. Let $\{\Phi_t; t \in \mathbb{R}_+\}$ be a convolution semigroup with generator L, i.e. $\mathrm{d}\Phi_t/\mathrm{d}t = L\Phi_t = \Phi_t L$. Suppose further that W is invertible. Then the Wigner distribution of Φ_t satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} W^*(\Phi_t) = \tilde{\rho}(L)^* W^*(\Phi_t)$$

with $\tilde{\rho}(X) = W^{-1} \circ \rho(X) \circ W$ and $\tilde{\rho}(X)^*$ defined by duality.

Proof. Differentiate

$$\langle \Phi_t, W(u)\rangle = \int u \,\mathrm{d}W^*(\Phi_t)$$

with respect to t; on the right-hand side we get $\int u\, \mathrm{d}(\mathrm{d}W^*(\Phi_t))/\mathrm{d}t$, while the left-hand side gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \Phi_t, W(u)\rangle = \langle L\Phi_t, W(u)\rangle = \langle \Phi_t, \rho(L)W(u)\rangle = \langle \Phi_t, W(\tilde{\rho}(L)u)\rangle$$
$$= \int \tilde{\rho}(L)u\,\mathrm{d}W^*(\Phi_t) = \int u\,\mathrm{d}(\tilde{\rho}(L)^* W^*(\Phi_t)).$$

4.1. Example: the braided line

Here we have only one variable, and the algebra is commutative, so the Weyl map is just $W : u^n \mapsto x^n$. The Fourier transform of the functional $\Phi = \sum_{n=0}^{\infty} a_n p^n$ is thus

$$g_\Phi(u) = \langle \Phi, \mathrm{e}^{\mathrm{i}ux}\rangle = \sum_{n=0}^{\infty} \frac{a_n q_n!(\mathrm{i}u)^n}{n!}$$

where we have assumed that the regularity conditions necessary for interchanging the limits are satisfied. For example, for $\Phi_a = \mathrm{e}^{ap}$, i.e. the functional determined by $\Phi_a(\mathrm{e}_q^{x|p}) = \mathrm{e}^{ap}$ (where $\mathrm{e}_q^{x|p} = \sum_{n=0}^{\infty} x^n p^n/q_n!$) we get

$$g_{\Phi_a}(u) = \sum_{n=0}^{\infty} \frac{q_n!(\mathrm{i}au)^n}{(n!)^2}.$$

We need $\tilde{\rho}(p)^*$ to be able to give the form of the Fokker–Planck equation. Because of the simple form of W we have $\tilde{\rho}(p) = \delta$. The adjoint of the q-difference operator is a multiple of the q-difference with q replaced by q^{-1},

$$\delta^* f(x) = -\frac{1}{q}\frac{f(q^{-1}x) - f(x)}{x(q^{-1}-1)} = -\frac{1}{q}\delta_{1/q} f(x)$$

so that the Wigner density $\eta_t(\mathrm{d}x) = \mathrm{d}W^*(\Phi_t)$ of the semigroup with generator $L = \sum c_n p^n$ satisfies

$$\partial_t \eta_t = \sum \frac{(-1)^n c_n}{q^n}\delta_{1/q}^n \eta_t.$$

4.2. Example: the braided plane

In order to determine W we calculate $(a_1 x + a_2 y)^N$, the coefficient of $a_1^n a_2^m$ is the image of $u_1^n u_2^m$. We get

$$W : u_1^n u_2^m \mapsto \frac{\left[{n+m \atop n}\right]}{\left({n+m \atop n}\right)} q x^n y^m$$

and thus

$$g_\Phi(v_1, v_2) = \sum_{n,m=0}^{\infty} \frac{\left[{n+m \atop n}\right]_q}{(n+m)!}\mathrm{i}^{n+m}\langle \Phi, x^n y^m\rangle v_1^n v_2^m.$$

For the Gauss functionals in the sense of Bernstein ($\Phi(x^n y^m) = z^n \delta_{m,0}$ or $z^m \delta_{n,0}$, see [8]) we get $g_\Phi(v) = \mathrm{e}^{\mathrm{i}zv_1}$ or $g_\Phi(v) = \mathrm{e}^{\mathrm{i}zv_2}$, i.e. $W^*(\Phi)$ is a Dirac mass in $(z, 0)$ or $(0, z)$.

To write down the Fokker–Planck equations for Wigner distributions we need the representation $\tilde{\rho}$. For the two generators we get

$$\tilde{\partial}_1 u_1^n u_2^m = n\frac{q_{n+m}}{n+m} u_1^{n-1} u_2^m$$
$$\tilde{\partial}_2 u_1^n u_2^m = mq^n\frac{q_{n+m}}{n+m} u_1^n u_2^{m-1}.$$

4.3. *Example: the q-affine group*

We get

$$W : u_1^n u_2^m \mapsto \sum_{\nu=0}^{n} \frac{n!m!}{(n+m)!} A_{n,\nu}^{n+m} \beta^{n-\nu} a^\nu b^m$$

where $A_{n\nu}^N$ are determined by $A_{n\nu}^N = 0$ if $n > N$ or $\nu > n$ or $n < 0$, $A_{00}^N = 1$, $A_{N\nu}^N = \delta_{N\nu}$, and

$$A_{n\nu}^{N+1} = A_{n-1,\nu-1}^N + (N+1-n)A_{n-1,\nu}^N + A_{n\nu}^N.$$

For the special case where the two lower indices are identical, we get $A_{nn}^N = \binom{N}{n}$.

The Weyl map and its inverse are characterized by

$$W : \mathrm{e}^{a_1u_1+a_2u_2} \mapsto \mathrm{e}^{a_1a+a_2b} = \mathrm{e}^{a_1a}\,\mathrm{e}^{((\mathrm{e}^{a_1\beta}-1)/a_1\beta)a_2b}$$
$$W^{-1} : \mathrm{e}^{b_1a}\,\mathrm{e}^{b_2b} = \mathrm{e}^{b_1a+(\beta b_1b_2/(\mathrm{e}^{b_1\beta}-1))b} \mapsto \mathrm{e}^{b_1u_1+(\beta b_1b_2/\mathrm{e}^{b_1\beta}-1))u_2}.$$

This allows us to write down $\tilde{\rho}$ and thus the Fokker–Planck equation for any Lévy process. For X we simply get $\tilde{\rho}(X) = \partial/\partial u_1$; the expression for $\tilde{\rho}(Y)$ is more complicated.

4.4. *Example: the q-Heisenberg–Weyl algebra*

The q-Heisenberg–Weyl algebra HW_q can be treated in the same way. Choose a, b, c as generators of HW_q, then the Weyl map is given by

$$W : u_1^n u_2^m u_3^r \mapsto \sum_{k=0}^{n\wedge r} \frac{n!m!r!D_{n,m,k}^{n+m+r}}{(n+m+r)!} a^{n-k} b^{m+k} c^{r-k}$$

where the coefficients $D_{n,r,k}^{n+m+r}$ are defined by

$$D_{n,r,k}^N = 0 \qquad \text{if } n < 0 \text{ or } r < 0 \text{ or } n+r > N \text{ or } k < 0 \text{ or } k > n \wedge r$$
$$D_{n,0,0}^N = \binom{N}{n} \qquad \text{if } 0 \leqslant n \leqslant N$$
$$D_{0,r,0}^N = \binom{N}{r} \qquad \text{if } 0 \leqslant r \leqslant N$$

and the recurrence relation

$$D_{n,r,k}^{N+1} = D_{n,r,k}^N + D_{n,r-1,k}^N + q^{k-r} D_{n-1,r,k}^N + (q^{-1})_{r-k+1}\left(1 - \frac{1}{q}\right) D_{n-1,r,k-1}^N.$$

5. Conclusion

We have given a relation between stochastic processes and evolution equations on quantum/braided groups, and given illustrations for several examples. Two types of evolution equation were considered. First, we let the generator of the process act via the dual representation on the quantum or braided group itself. Solutions to these evolution equations are given in terms of Appell systems. We have shown in several examples how they can be calculated. To find the second kind of evolution equation, we have associated Wigner-type distributions to the processes and functionals. These distributions are distributions on the undeformed space, and we thus get evolution equations for ordinary functions. If we replace the Weyl map by other maps, for example, those corresponding to normal ordering (‘Wick’ or ‘anti-Wick’) or to q-exponentials, then we can hope to find a simpler expression for $\tilde{\rho}$ and thus for the Fokker–Planck equation, but the relation between the moments of the functionals and their ‘Wigner’ distributions becomes more complicated.

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